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A Mixed Formulation in Conjunction with the Penalization Method for Solving the Bilaplacian Problem with Obstacle Type Constraints

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ABSTRACT

In this paper we consider a mixed formulation for study of the bilaplacian problem with obstacle constraints in conjunction with the penalization method. The idea is based on the decomposition of the bilaplacian operator into two coupled laplacians and of course by choosing suitable spaces. The numerical advantage is to allow the solution of coupled systems with nice matrices having the M-matrix property. Moreover, the computed solution of the problem requires less execution time with respect to the discrete system of the bilaplacian problem. For simulation, we test the efficiency of a variety of iterative relaxation methods and discuss their numerical performances.

Keywords: Mixed formulation, Unilateral constraint, Bilaplacian problem, Method of penalization.

1. Introduction

The obstacle problems are a very important class of nonlinear problems which have been widely studied mainly for 2^{nd} order operators from both theoretical and numerical point of views by **?**, **?**, **?**, **?**, **??**. Early research had been concerned with the study of the 2^{nd} order problems with constraints. We refer, particularly, to the book of **?** and **?**.

In **?**, the author stated and established a general result of existence and uniqueness of solution for the stationary and the evolutive problems. These problems are, generally, encountered in various domains such as mechanics, physics, economics, optimal control, etc., and correspond to strongly nonlinear problems. The nonlinearity is due to the nonlinear nature of the constraints imposed on the solution inside the domain Ω or on the boundary $\partial\Omega$.

In **?**, we can find a unified approach for both the classical obstacle and Signorini problems. The main difference between the two distinct problems resides in the choice of a particular functional defined by an integral either on the boundary for the Signorini problem or in the domain for the obstacle problem. We are now ready to state the model problem:
Find $u \in V$ such that

$$(P) \begin{cases} Au = f \quad \text{in } \Omega, & (a) \\ u \geq q \quad \text{in } \Omega, & (b) \\ (Au - f)(u - q) = 0 \quad \text{in } \Omega, & (c) \\ \text{initial conditions plus boundary conditions on } \partial\Omega, & (d) \end{cases}$$

where A is, generally, an elliptic or parabolic operator; f is a given function belonging to some space which ensures the regularity of the solution u; q represents the unilateral constraint and V is a Hilbert space. The equation (c) is called the complementarity condition.

The second order problem is still a subject of several studies, we cite, for example, a recent work (see **?**), where the authors have studied a $3D$ problem with unilateral constraints on the boundary arising in fluid mechanics and proposed an implicit scheme for the discretization of the time dependent part of the operator. Hence, the problem is reduced to the solution of a sequence of stationary problems by the domain decomposition method using an asynchronous parallel iterative algorithm.

Fourth order problems have received much less attention but however in the last two decades they regained a renewed interest because of their important applications (see **?**, **?**, **?**, **?**, **?**, **?** and **?**). Accordingly, a great importance has been given to the construction of iterative numerical algorithms which take efficiently into account the constraint (see **?**, **?** and **?**).

Note that these problems are considered as ill-posed problems (see **?**. However the theorem of maximum principle cannot be applied to them contrary to 2^{nd} order problems. We cite some relevant references; in **?**, the authors have reduced the ill-posed problem to an optimal control problem and established necessary conditions of optimality. In **?**, the obtained results show that the discretization of the obstacle problem for a thin elastic plate by the mixed finite element method of a variational formulation can be obtained directly from the hypersphere inequality and led to two optimal convergence rates. For our part, we are interested in the study of the bilaplacian problem subject to unilateral constraints using a technique of mixed formulation due to **?**. We can find some properties of induced functional spaces concerning this technique (see **?**).

The numerical advantage is that it requires only the use of C^0 finite elements, contrary to the initial formulation using the bilaplacian which requires C^1-continuity. The idea is based on the decomposition of the bilaplacian operator into two coupled laplacians and of course by choosing suitable spaces. For this purpose, let u be a solution of the bilaplacian problem and introduce the function ϕ, defined by $\Delta u = -\phi$, where $\phi \in L_2(\Omega)$ and $\Delta\phi \in H^{-1}(\Omega)$ (which is useful because Δu represents the bending moments in elasticity and the vorticity in hydrodynamics). Several authors have studied this decomposition technique more specially for the bilaplacian equation without obstacle (see **?**, **?** and **?**). For example, **?** have studied the mixed finite element method which gives not only the approximation u_h of u but also the approximation ϕ_h of Δu, where ϕ_h and u_h are the approximations of ϕ and u respectively. In **?**, the authors have introduced the polynomials of degree k, with $k \geq 2$ for the particular case of a polygonal domain and obtained the following convergence order

$$\|\phi - \phi_h\|_{0,\Omega} + |u - u_h|_{1,\Omega} = O(h^{k-1}),$$

where $\|.\|_{0,\Omega}$ and $|.|_{1,\Omega}$ are classical norms [1].

? proposed a method which preserves the property of the bilaplacian equation (without obstacle) by formulating it as an operator on opposite boundaries. This method allows the use of conjugate gradient technique which requires a

[1] $\forall v \in L^2(\Omega)$, $\|v\|_{0,\Omega} = \left(\int_\Omega |v|^2\, d\Omega\right)^{1/2}$ and $\forall v \in H^1(\Omega)$, $|v|_{1,\Omega} = \left(\int_\Omega |\nabla v|^2\, d\Omega\right)^{1/2}$.

solution of two Laplace-Dirichlet problems by iteration (see **?**). The obtained results confirm the efficiency of the method, especially regarding the computation of ϕ in the vicinity of the boundary domain.

In this study, we look for vertical displacements u along the oz axis of a square elastic plate Ω perfectly clamped on its edges (or with simply supported boundary conditions), subject to a force f perpendicular to the thin elastic plate and displaced by a rigid obstacle. For this, we propose weak variational formations and state the advantages and shortcomings of each of them as well as their approximations by an appropriate finite element method. On the one hand, the mixed formulation is more suitable than that of the above problem (P) since it reduces the 4^{th} order problem to two 2^{nd} order and hence requiring less regularity of the solution. On the other hand, it has a numerical advantage since it allows to solve coupled systems with nice matrices having the M-matrix property. Moreover, the computed solution of the problem requires less execution time with respect to the discrete system of the bilaplacian problem. So, the goal of this paper is to combine the mixed formulation with the penalization method. The latter permits the removal of the constraints by incorporating them into the functional. Afterwards, we complete this study by numerical simulations, testing the efficiency of the iterative relaxation methods.

The outline of the paper is as follows. Section 2 describes the problem, the variational formulations and the existence and uniqueness of the solution to the problem. Section 3 introduces the decomposition technique and gives the existence and uniqueness of the solution to the coupled sub-problems. Section 4 applies the penalization method to the coupled sub-problems. Section 5 studies the discrete analogues and the last section displays the numerical results.

2. Description of the problem

In elasticity, the problem describes, in $2D$ space, the vertical displacements u along the oz axis of a thin elastic plate Ω perfectly clamped on its edges (or with simply supported boundary condition), subject to a force f perpendicular to this plate and in contact with a rigid obstacle q (see Figure 1).
Let Ω be a bounded open set of R^n with boundary $\partial\Omega$ sufficiently regular and V a Hilbert space. We seek a function $u \in V$ such that

$$\Delta^2 u = f \qquad \text{in } \Omega, \tag{1}$$

$$u \geq q \qquad \text{in } \Omega, \tag{2}$$

$$(\Delta^2 u - f)(u - q) = 0 \qquad \text{in } \Omega, \tag{3}$$

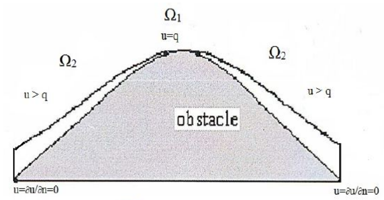

Figure 1: The clamped thin elastic obstacle problem

with clamped boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{4}$$

or with simply supported boundary conditions

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega, \tag{5}$$

where the bilaplacian operator Δ^2 is defined by

$$\Delta^2 = \Delta(\Delta) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

the given function f belongs to $L^2(\Omega) \cap H^4(\Omega)$ and n is an external normal vector to Ω; here $V = H_0^4(\Omega)^2$.

Note that the equation (**??**) represents the complementarity condition and the problem (**??**)-(**??**) is considered as a free boundary problem which can be interpreted as follows.

The solution u divides the domain Ω into two subdomains Ω_1 and Ω_2 (see the Figure 1) as shown below

$$\Omega = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 = \{x \in \Omega / \ u = q\}, \quad \Omega_2 = \{x \in \Omega / \ u > q\} \text{ and } S = \partial\overline{\Omega}_1 \cap \partial\overline{\Omega}_2$$

with S is a priori an unknown interface separating Ω_1 and Ω_2.

[2] $H_0^4(\Omega) = \left\{v : \Omega \subset \mathbb{R}^n \to \mathbb{R}, \ \frac{\partial^k v}{\partial x_{i,1} \ldots \partial x_{i,k}} \in L^2(\Omega), \ \forall k = \overline{0,4}\right\} \cap \left\{v/v = \frac{\partial v}{\partial n} = 0 \,\text{on}\, \partial\Omega\right\}$

In the sequel of the analytical study, we will use mainly the boundary conditions (**??**). Note that a study of the problem (**??**)-(**??**) with the boundary conditions (**??**) can be found in **?**. Knowing that the problem (**??**)-(**??**) can be written as an optimization problem. For this, let us consider the following problem

$$J(u) = \inf_{v \in K} J(v), \tag{6}$$

where

$$J(v) = \frac{1}{2} \int_{\Omega} |\Delta v|^2 \, d\Omega - \int_{\Omega} f v d\Omega, \tag{7}$$

with

$$K = \left\{ v \in H_0^2(\Omega) / v \geq q \right\}^3. \tag{8}$$

Setting

$$a(u, v) = \int_{\Omega} \Delta u . \Delta v d\Omega, \tag{9}$$

where the bilinear form $a(.,.)$ is none other than the scalar product whose $H^2(\Omega)$ norm comes from (**??**).

[3] $H_0^2(\Omega) = \left\{ v : \Omega \subset \mathbb{R}^n \to \mathbb{R},\ \frac{\partial^k v}{\partial x_{i,1} \dots \partial x_{i,k}} \in L^2(\Omega),\ \forall k = \overline{0,2} \right\} \cap \left\{ v / v = \frac{\partial v}{\partial n} = 0 \,\text{on}\, \partial\Omega \right\}$

We know that the Sobolev spaces $H^2(\Omega)$ and $H_0^2(\Omega)$ (the space $H_0^2(\Omega)$ is a closed subspace of the space $H^2(\Omega)$) are Hilbert spaces for the norm $\|.\|_{2,\Omega}$ [4] (see **?**).

Let us also set

$$F(v) = (f, v) = \int_\Omega f.vd\Omega,$$

where $(.,.)$ denotes the inner product defined in $L^2(\Omega)$.
Thus the functional J can be written as follows

$$J(u) = \frac{1}{2}a(u,u) - F(u).$$

So, the problem (**??**)-(**??**) is equivalent to

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) \leq J(v), \;\; \forall v \in K. \end{cases} \tag{10}$$

If the minimization problem (**??**) has one solution satisfying the equations (**??**)-(**??**), then the Euler inequality applied to the problem (**??**) leads to the classical equivalent variational formulation of the boundary value problem

$$\begin{cases} \text{Find the solution } u \in K \text{ such that} \\ a(u, v-u) \geq F(v-u) \qquad \forall v \in K. \end{cases} \tag{11}$$

Let V be a reflexive Banach space; the form $a(u,v)$ is continuous and linear in v but nonlinear in u, i.e., $a(u,v) = (Au, v)$, where the operator $A : V \to V$. Thus, the problem (**??**) can be written as follows

$$\begin{cases} \text{Find the solution } u \in K \text{ such that} \\ (Au, v-u) \geq F(v-u) \qquad \forall v \in K. \end{cases} \tag{12}$$

The existence and uniqueness of the solution of problem (**??**) is ensured since the bilinear form $a(u,v) = \int_\Omega \Delta u.\Delta v d\Omega$ is continuous and V-elliptic (here $V = H^2(\Omega)$) and the linear form $F(v) = \int_\Omega f.vd\Omega$ is also continuous. Then, according to Lions-Stampacchia theorem, the variational inequality (**??**) admits a unique solution $u \in H_0^2(\Omega)$.

[4] $\forall v \in H_0^2(\Omega) : \|v\|_{2,\Omega} = \left[\|v\|_{0,\Omega}^2 + \sum_{i=1}^{2} \left\| \frac{\partial v}{\partial x_i} \right\|_{0,\Omega}^2 + \sum_{i.j=1}^{2} \left\| \frac{\partial v^2}{\partial x_i \partial x_j} \right\|_{0,\Omega}^2 \right]^{\frac{1}{2}}$

The existence and uniqueness of the solution for the problem (**??**) is also ensured if A is coercive, i.e.,

$$\lim_{\|v\|\to\infty} \frac{(Av, v)}{\|v\|} = \infty,$$

and strictly monotone, i.e.,

$$(Au - Av, v - u) > 0 \text{ for any } u, v \in V.$$

3. A Mixed formulation for the problem (??)-(??)

? proposed a method based on a variational formulation called mixed formulation (decomposition method) in order to use numerically $P1$ Lagrange FEM. For our part, we adapt this technique to a problem with unilateral constraints. As said previously, this technique is based on the decomposition of the problem (**??**)- (**??**) into two coupled sub-problems. To do this, let u be a solution of the problem (**??**)-(**??**) and introduce now the function ϕ as follows:

$$\triangle u = -\phi. \tag{13}$$

Substituting (**??**) into the problem (**??**)-(**??**) and introducing the appropriate variational framework, i.e., ϕ in $L^2(\Omega)$ with $\triangle\phi$ in $H^{-1}(\Omega)$ and u in $H_0^1(\Omega)$, this new framework will enable us to define an approximate method of class C^0.

If u is a solution of problem (**??**)-(**??**) then a pair $(\phi, u) \in H^2(\Omega) \times H_0^2(\Omega)$ is a solution of the following continuous problem:

$$(P_{mix}) \qquad \begin{cases} -\Delta\phi = f & \text{in } \Omega, \\ -\triangle u = \phi & \text{in } \Omega, \\ u \geq q \quad \text{in } \Omega, \\ (\Delta\phi + f)(\triangle u + \phi)(u - q) = 0 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

To give a variational inequality corresponding to the continuous problem (P_{mix}), let (Ψ, v) be a pair of test functions belonging to $H_0^1(\Omega) \times H^1(\Omega)$, where the function Ψ will be test functions associated to ϕ, and v to the second variable u. Note that the values of u do not intervene in the formulation (**??**,

i) given below, in order to preserve their memory in the continuous problem (P_{mix}). Thus, we can introduce the properties that are suitable in the functional space, for which the function Ψ will be generic. The boundary conditions on $\partial\Omega$ are replaced respectively in the systems of equations and in the inequality of the problem (P_{mix}) by choosing as functional space $H^1(\Omega) \times H_0^1(\Omega)$ for the unknowns (ϕ, u) and $H_0^1(\Omega) \times H^1(\Omega)$ for the pair of test functions (Ψ, v).

Thus, the coupled variational inequalities corresponding to problem (P_{mix}) can be written as follows:

$$\left\{\begin{array}{lll} \text{Find the couple } (\phi, u) \in H^1(\Omega) \times K_1 \text{ such that} & & \\ a_1(\phi, \Psi) = L_f(\Psi), & \forall \Psi \in H_0^1(\Omega) & (i) \\ a_2(u, v-u) \geq L_\phi(v-u), & \forall v \in K_1 & (ii) \end{array}\right. \tag{14}$$

where

$$a_1(\phi, \Psi) = \int_\Omega \nabla\phi.\nabla\Psi d\Omega, \qquad L_f(\Psi) = \int_\Omega f.\Psi d\Omega,$$

$$a_2(u, v) = \int_\Omega \nabla u.\nabla v d\Omega, \qquad L_\phi(v) = \int_\Omega \phi.v d\Omega,$$

and

$$K_1 = \{v \in V_1/\ v > q\}, \text{ with } V_1 = H_0^1(\Omega) \cap \left\{v/\frac{\partial v}{\partial n} = 0 \,\text{on}\ \partial\Omega\right\}.$$

For the existence and uniqueness of solution for the coupled variational inequalities (**??**), we state respectively the following theorems.

Theorem 3.1. *Let V $(V = H^1(\Omega))$ be a Hilbert space, the bilinear form $a_1(.,.)$ is continuous and V-elliptic and $L_f(.)$ continuous in $H^1(\Omega)$. Then, according to Lax-Milgram theorem, the variational equation (**??**, i) admits a unique solution ϕ belonging to $H^1(\Omega)$.*

In the same way, we state the following theorem for the variational inequality (**??**, ii).

Theorem 3.2. *(**?**) Let V_1 be a Hilbert space and K_1 a nonempty convex and closed subset of V_1, the bilinear form $a_2(.,.)$ is continuous and V-elliptic and let $L_\phi(.)$ be continuous in V_1. Then, according to Lions-Stampacchia theorem, the problem (**??**, ii) admits a unique solution u belonging to V_1.*

4. Penalization method

In this section, we introduce the penalization method (for more details, see ?, and ?) for the problems (??), (??) and (??). To do this, we recall the following definitions.

Definition 4.1. *The functional $f : V \to \overline{R}$ is proper if $f(x) > -\infty$ for all $x \in V$ and f does not identically equal to infinity.*

Definition 4.2. *The functional $f : V \to \overline{R}$ is lower semi-continuous (l.s.c.) at x, if for every sequence $\left\{x^j\right\}_{j\geq 0}$ converging to x, we have $f(x) \leq \lim\limits_{j\to\infty} \inf f(x^j)$.*

Let us introduce the functional

$$\gamma : V \to R \cup \{\infty\},$$

which verifies the following hypotheses

$$(H1) \qquad \begin{cases} \gamma \text{ is convex, proper and l.s.c.,} \\ \gamma(v) \geq 0, \forall v \in V \text{ (here } V = H_0^2(\Omega)), \\ \gamma(v) = 0 \text{ if and only if } v \in K. \end{cases}$$

For every $\varepsilon > 0$, the penalization of the problem (??) is

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^2(\Omega) \text{ such that} \\ a(u_\varepsilon, v - u_\varepsilon) + \frac{1}{\varepsilon}\gamma(v) - \frac{1}{\varepsilon}\gamma(u_\varepsilon) \geq F(v - u_\varepsilon), \quad \text{for any } v \in H_0^2(\Omega). \end{cases} \tag{15}$$

Theorem 4.1. *(?) Under the hypotheses $(H1)$, the problem (??) admits a unique solution for any fixed $\varepsilon > 0$.*

Theorem 4.2. *(?) Let u be solution of the problem (??) and u_ε be solution of the problem (??), then*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\| = 0,$$
$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\gamma(u_\varepsilon) = 0.$$

To penalize the problem (??), let us also introduce the operator $\beta : V \to V'$ (V' dual space of V) which has the following properties:
- β is Lipschitz continuous (globally or locally);

- $\ker(\beta) = K$, i.e., $\beta(v) = 0$ if and only if $v \in K$;
- β is monotone.

For any $\varepsilon > 0$, the penalized problem associated to (**??**) is written as follows

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^2(\Omega) \text{ such that} \\ (Au_\varepsilon, v) + \frac{1}{\varepsilon}(\beta(u_\varepsilon), v) \geq F(v), \quad \text{for any } v \in H_0^2(\Omega). \end{cases} \tag{16}$$

To penalize the problem (**??**), in similar way, we replace in the hypotheses $(H1)$ the space V by $H_0^1(\Omega)$ and the subset K by K_1, so that the problem will be given by

$$\begin{cases} \text{Find } (\phi, u_\varepsilon) \in H^1(\Omega) \times H_0^1(\Omega) \text{ such that} & \\ a_1(\phi_\varepsilon, \Psi) = L_f(\Psi), \qquad \forall \Psi \in H^1(\Omega), & (i_\varepsilon) \\ a_2(u_\varepsilon, v - u_\varepsilon) + \frac{1}{\varepsilon}\gamma(v) - \frac{1}{\varepsilon}\gamma(v_\varepsilon) = L_{\phi_\varepsilon}(v - u_\varepsilon), \ \forall v \in H_0^1(\Omega). & (ii_\varepsilon) \end{cases} \tag{17}$$

According to theorem 6, problem (**??**) admits a unique solution for every fixed $\varepsilon > 0$. To show the strong convergence, we apply theorem 7 to problem (**??**).

Penalization algorithm
The penalization algorithm for the problem (**??**) is given as follows

$S0$: choosing u^0, ε_1
$S1$: put $k = 1$,
$S2$: find $u^k \in V$ solution of problem (**??**), by taking u^{k-1} as an initial vector,
$S3$: give a stopping criterion, if u^k is "satisfactory" stop,
otherwise put $k := k + 1$, choose $\varepsilon_{k+1} < \varepsilon_k$ and return to $S2$.

5. Discrete analogues of problems (??) and (??)

Let $\{V_{1,h}\}_{h>0}$ be a family of closed subspaces of V_1 (finite dimensional). Let $\{K_{1,h}\}_{h>0}$ be also a family of convex, closed and not empty subsets of V_1 such that $K_{1,h} \subset V_{1,h}$ for every h. The family $\{K_{1,h}\}_{h>0}$ approximates the set K_1, i.e., $\lim\limits_{h \to 0} K_{1,h} = K_1$, if the following conditions are satisfied:
if $\{v_h\}_{h>0}$ is a sequence such that $v_h \in K_{1,h}$, for every $h > 0$ and $v_h \to v$ weakly in V_1, then $v \in K_1$; - for any $v \in K_1$, there exists a sequence $\{v_h\}_{h>0}$ such that $v_h \in K_{1,h}$ for any $h > 0$ and $v_h \to v$ strongly in V_1.

Thus, we use the FEM discretization. To this end, we introduce a grid of equidistant nodes as follows: $x_i = ih$, for $i = 0, 1, ..., N$ with $h = 1/N$ and

$y_j = jk$, for $j = 0, 1, ..., M$ with $k = 1/M$ (for simplicity we take $N = M$) . The subspace $V_{1,h} \subset V_1$ is defined by

$$V_{1,h} = \left\{ v_h \in H^1(\Omega) \cap C(\Omega) / \quad v/_\Gamma \in P_1 \quad \text{on each triangle} \right\}$$

and

$$\widetilde{V}_{1,h} = \{ v_h \in V_{1,h} \text{ such that } \quad v_h = \tfrac{\partial v_h}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \}.$$

and the discrete convex set:

$$K_{1,h} = \{ v_h \in \widetilde{V}_{1,h} / v_h(x_i, y_j) \geqslant q(x_i, y_j) \text{ in } \Omega, \text{with } (x_i, y_j) \text{ the nodal points} \}$$

The discretization of problems (**??**) is given by

$$\begin{cases} \text{Find the couple } (\phi_h, u_h) \in V_{1,h} \times \widetilde{V}_{1,h} \text{ such that} & \\ a_1(\phi_h, \Psi_h) = L_f(\Psi_h), \qquad \forall \Psi_h \in \widetilde{V}_{1,h}(\Omega), & (i_h) \\ a_2(u_h, v_h - u_h) \geq L_\phi(v_h - u_h), \forall v_h \in K_{1,h}. & (ii_h) \end{cases} \tag{18}$$

Now, constructing the discrete analogue of the penalized problem (**??**), any $v_h \in V_{1,h}$ is represented by the vector of its coefficients which belong to R^{N+1}. We denote by respectively $v_{i,j}$ and $q_{i,j}$ the $v_h(x_i, y_j)$ and $q(x_i, y_j)$, hence, $K_{1,h}$ may be reduced to $R_{1,h} \subset R^{N+1}$.

For $i, j = \overline{0, N-1}$, $R_{1,h}$ is defined by

$$R_{1,h} = \left\{ (v_{i,j} \in (R^{N+1})^2 / \ v_{i,j} \mid_{\partial\Omega} = \frac{\partial v_{i,j}}{\partial n} \mid_{\partial\Omega} = 0 \text{ and } v_{i,j} \geq q_{i,j} \right\}. \tag{19}$$

Let us now introduce the function

$$T : R^{N+1} \to R,$$

$$T(v) = \frac{1}{2} \sum_{i,j=0}^{n} ((v_{i,j} - q(x_i, y_j))^-)^2 \text{ for } \{v_{i,j}\}_{i,j=0}^{N},$$

where $(v_{i,j} - q(x_i, y_j))^-$ is the negative part of $(v_{i,j} - q(x_i, y_j))$ for $i, j = 0, ..., N$, and defined by

$$(v_{i,j} - q(x_i, y_j))^- = \max_{0 \leq i,j \leq N} \{0, (q(x_i, y_j) - v_{i,j})\}.$$

For $i, j = 0, 1, ...N$, we have

$$\frac{\partial T}{\partial v_{i,j}}(v) = (v_{i,j} - q(x_i, y_j)^- \ \frac{\partial}{\partial v_{i,j}}((v_{i,j} - q(x_i, y_j))^-) = -(v_{i,j} - q(x_i, y_j)^-.$$

We define the operator $\eta : R^{N+1} \to R^{N+1}$ as follows:

$$\eta(v) = \nabla T(v), \ \forall v \in R^{N+1}. \tag{20}$$

Lemma 5.1. *(?) The operator $\eta : R^{N+1} \to R^{N+1}$ defined by (??) verifies the following properties:*

- *η is Lipschitz continuous,*
- *$\eta(v) = 0$ if and only if $v \in R_{1,h}$,*
- *η is monotone.*

The discrete analogue of the penalized problems (??) are written as follows

$$\begin{cases} \text{Find } (\phi_{\varepsilon,h}, u_{\varepsilon,h}) \in V_{1,h} \times \widetilde{V}_{1,h} \text{ such that} \\ A_{1,h}\phi_{\varepsilon,h} = f_h & (i_{\varepsilon,h}) \\ A_{2,h}u_{\varepsilon,h} + \frac{1}{\varepsilon}\eta(u_{\varepsilon,h}) = \phi_{\varepsilon,h} & (ii_{\varepsilon,h}) \end{cases} \tag{21}$$

where $A_{1,h}$ and $A_{2,h}$ are respectively the discretization matrices of the problems (??, i) and (??, ii).

For simplicity of notation, we drop the indices ε and h everywhere in the sequel and replace the indices i, j in the terms $u_{i,j}$ and $q_{i,j}$ by choosing a single indice. For $i = \overline{0, N-1}$, the systems (??) become

$$\begin{cases} \sum\limits_{l=0}^{N-1} a_{i,l}^{(1)}\phi_l = f_i & (i_{\varepsilon}^{'}) \\ \sum\limits_{l=0}^{N-1} a_{i,l}^{(2)}u_l - \frac{1}{\varepsilon}((u_i - q_i)^- = \phi_i, & (ii_{\varepsilon}^{'}) \end{cases} \tag{22}$$

where $(a_{i,l}^{(1)})_{i,l=0,\dots,N-1}$ and $(a_{i,l}^{(2)})_{i,l=0,\dots,N-1}$ are respectively the coefficients of the discretization matrices $A_{1,h}$ and $A_{2,h}$.

The linear systems (??, $i_{\varepsilon}^{'}$) can be solved, for example, by Gauss-Seidel iterative method and the nonlinear system (??, $ii_{\varepsilon}^{'}$) by S.O.R method.

For $l = 0, ..., N-1$, there exist two cases:
1. $u_i \geq q_i$, then $(u_i - q_i)^- = 0$ for $i = \overline{0, N-1}$ and the system (**??** - $ii_\varepsilon^{'}$) becomes

$$\begin{cases} \sum\limits_{l=0}^{N-1} a_{i,l}^{(1)} \phi_l = f_i, \\ \sum\limits_{l=0}^{N-1} a_{i,l}^{(2)} u_l = \phi_i. \end{cases} \tag{23}$$

We solve the first system (**??**) by Gauss-Seidel method and the second system by the S.O.R method. At the k^{th} iteration, for $i = \overline{0, N-1}$, we have

$$\begin{cases} \phi_i^{k+1} = -\frac{1}{a_{i,i}^{(1)}} \left(f_i + \sum\limits_{l=0}^{i-1} a_{i,l}^{(1)} \phi^{k+1} + \sum\limits_{l=i+1}^{N-1} a_{i,l}^{(1)} \phi^k \right), \\ u_i^{k+1} = (1-\omega) u_i^k - \frac{\omega}{a_{i,i}^{(2)}} \left(\phi_i^{k+1} + \sum\limits_{l=0}^{i-1} a_{i,l}^{(2)} u_l^{k+1} + \sum\limits_{l=i+1}^{N-1} a_{i,l}^{(2)} u_l^k \right), \end{cases} \tag{24}$$

where ω is the relaxation parameter.

2. $u_i < q_i$ then $(u_i - q_i)^- = q_i - u_i$ for $i = \overline{0, N-1}$ and the coupled systems (**??**) become

$$\begin{cases} \sum\limits_{l=0}^{N} a_{i,l}^{(1)} \phi_l = f_i, \\ \sum\limits_{l=0}^{N} a_{i,l}^{(2)} u_l - \frac{1}{\varepsilon}(q_i - u_i) = \phi_i. \end{cases} \tag{25}$$

At the k^{th} iteration, for $i = \overline{0, N-1}$, we have

$$\begin{cases} \phi_i^{k+1} = -\frac{1}{a_{i,i}^{(1)}} \left(f_i + \sum\limits_{l=0}^{i-1} a_{i,l}^{(1)} \phi^{k+1} + \sum\limits_{l=i+1}^{N-1} a_{i,l}^{(1)} \phi^k \right), \\ u_i^{k+1} = (1-\omega) u_i^k - \frac{\omega}{\widetilde{a}_{i,i}} \left(\widetilde{\phi}_i^{k+1} + \sum\limits_{l=0}^{i-1} a_{i,l}^{(2)} u_l^{k+1} + \sum\limits_{l=i+1}^{N-1} a_{i,l}^{(2)} u_l^k \right), \end{cases} \tag{26}$$

where $\widetilde{a}_{i,i} = a_{i,i}^{(2)} + \frac{1}{\varepsilon}$ and $\widetilde{\phi}_i = \phi_i + \frac{1}{\varepsilon} q_i$.

6. Numerical examples

To illustrate the above described study, we propose two examples.
Example 1. We take $f \equiv 0$ and

$$q(x,y) = \begin{cases} -\frac{2}{5} + \sqrt{\frac{1}{4} - (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2}, \text{ if } \ (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 \leq \frac{1}{3}. \\ 0, \quad \text{otherwise.} \end{cases}$$

Also, we take $\Omega = (0,1)^2$, where the right hand side f of the equation (**??**) is computed by using the below exact solution

$$u(x,y) = (e^x + (x+1)e^y)x^2y^2(1-x)^2(1-y)^2,$$

which satisfies the clamped boundary condition $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

Let's carry out a number of tests to determine the best choice of the relaxation parameter ω which makes it possible to reduce the iteration number and the execution time of the algorithm (**??**)-(**??**).

For this, we use the penalization parameter $\varepsilon = 10^{-4}$ and the various norms $\frac{|u-u_h|_{0,\Omega}}{|u|_{0,\Omega}}$, $\frac{|u-u_h|_{1,\Omega}}{|u|_{1,\Omega}}$ and $\frac{|\phi-\phi_h|_{0,\Omega}}{|\phi|_{0,\Omega}}$.

We display the results, for $h = 1/64$, in Table 1 and in Table 2, respectively.

ω	1.6	1.7	1.8	1.9
Iterations	128	128	84	89

Table 1. Tests for several values of ω.

We remark that the most appropriate parameter is $\omega = 1.8$. We display, in Table 2, the discretization errors along with the rates of convergence in various norms, for example, at the some nodes: for N1 to N9 (see Figure 2).

	$\frac{\lvert u-u_h\rvert_{0,\Omega}}{\lvert u\rvert_{0,\Omega}}$	$\frac{\lvert u-u_h\rvert_{1,\Omega}}{\lvert u\rvert_{1,\Omega}}$	$\frac{\lvert\phi-\phi_h\rvert_{0,\Omega}}{\lvert\phi\rvert_{0,\Omega}}$
N1	5.43290e-01	6.32693e-01	6.32041e-01
N2	3.26972e-01	4.01635e-01	5.16879e-01
N3	1.30302e-01	1.89139e-01	3.34937e-01
N4	3.99107e-02	8.32646e-02	1.88319e-01
N5	1.08809e-02	3.88438e-02	9.92016e-02
N6	2.82773e-03	1.89646e-02	5.08074e-02
N7	7.19891e-04	9.41839e-03	2.56967e-02
N8	1.81559e-04	4.70081e-03	1.29204e-02
N9	1.49139e-04	3.99039e-03	1.01139e-02

Table 2. Discretization errors in different norms for the clamped boundary condition problem

In same manner, we can perform similar tests for the parameter of penalization. According to the above results, we can see that the method is convergent and the precision of the solution is acceptable for practical purposes.

Example 2. In this example, we apply the previously described method to problem (**??**)-(**??**) with boundary conditions (**??**), i.e., a square plate simply supported on $\partial\Omega$. We take an example (see **?**) and we adapt it to our situation. The function f is given by

$$f = \frac{k(x,y)}{D}$$

where k is a supporting charge and D given by

$$D = \frac{E.ep^3}{12.(1-\nu^2)},$$

with ep the thickness of the plate (mm), ν the Poisson's coefficient (dimensionless), E the Young's modulus (DaN/mm^2) and L the plate length and width (mm).

To solve the system (**??**)-(**??**), we use Gauss-Seidel method (i.e., the relaxation parameter $\omega = 1$). Let's perform a number of experimental tests to determine the best choice of the penalization parameter ε. For simulation, we retain the same obstacle q as in example 1 and take the same data such that $k = 10$, $ep = 1$, $\nu = 0.3$, $E = 21000$ and $L = 20$. We choose a tolerance $tol = 10^{-10}$ and a discretization step $h = 1/40$. Now, we display in Table 3, the iteration number (iter) and solutions, for example, at the some nodes: for N1 to N9 (see

Figure 2).

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
iter	161	160	107	161
N1	0.35148241980	0.35145024454	0.35144800092	0.65084870177
N2	0.65091791811	0.65085372995	0.65084924526	0.87362326465
N3	0.87372465890	0.87363063270	0.87362404760	0.65084570177
N4	0.65091791811	0.65085372995	0.65084924526	1.20942564367
N5	1.20956577721	1.20943583177	1.20942672684	1.62605721021
N6	1.62626743670	1.62607251321	1.62605880566	0.87362326465
N7	0.87372465890	0.87360632704	0.87362404760	1.62605721021
N8	1.62626743674	1.62607251321	1.62605880566	1.62605721021
N9	2.18626575728	2.18595974290	2.18593808993	2.18593563510
cpu	$4.130524E + 01$	$4.017213E + 01$	$4.13552E + 01$	$4.2505531E + 01$

Table 3. Solutions at nodes N1 to N9 for different values of ε

We can see from the last column of the above table that the meshed is divergent for $\varepsilon = 10^{-4}$. Now we look for the numerical solution at the same nodes from N1 to N9 (see Table 4) by varying the discretization step h.

h	$h = 1/10$	$h = 1/20$	$h = 1/40$
iter	156	520	161
N1	0.35461036999	0.35192505960	0.35145024454
N2	0.65591825902	0.65153232928	0.65085372995
N3	0.87996516152	0.87438361745	087363063270
N4	0.65591825902	0.65153232928	0.65085372995
N5	1.21748764448	1.21034203389	1.20943583177
N6	1.63639174975	1.62704232558	1.62607251321
N7	0.87996516152	0.87438361745	0.87360632704
N8	1.63639174975	1.62704232558	1.62607251321
N9	2.20019244146	2.18704648324	2.19855974290
cpu	1.76769941	10.4258	$4.0172133E + 01$

Table 4. Solutions at nodes N1 to N9 for different step h

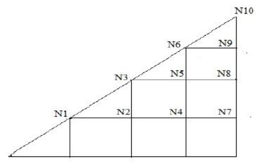

Figure 2: The nodes N_1 to N_9

7. Conclusion

In this work, we have combined the mixed formulation with the classical penalization method for a class of free boundary problems for which the obstacle problem is a representative. The main interest of the mixed formulation is to reduce the 4^{th} order problem to two 2^{nd} order problems. The latter problems require less regularity to the solution and then lead to the solution of two coupled classical laplacian sub-problems with Dirichlet boundary conditions under unilateral constraints. From the numerical point of view, the reduced problems allow to solve coupled systems of small sizes with nice matrices having the M-matrix property. Hence, the computed solutions of the reduced problems require less execution time comparatively to the discrete system associated to the bilaplacian problem. Moreover the coupled problems make use only of C^0-continuity ($P1$ Lagrange finite elements) contrary to the bilapacian problem which requires C^1-continuity. However, there is also a possibility to implement a parallel asynchronous algorithm. Through this study, we wanted to test the effectiveness of this technique in order to apply it to other more complex problems than the one presented in this study such as hyperbolic or parabolic type problems arising in hydrodynamics, finance or optimal control with state and/or control constraints.

Competing Interests

The authors declare that they have no competing interests.

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